Category 19
Pages 8
Code 1
CR # 85248
N67-30064

Differential-Boundary Equations
and
Associated Boundary Value Problems

by

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1. Differential-boundary equations first appear as adjoints in an article by Hilb [6] in 1911. With the exception of Hilb's student Betschler [1] they do not seem to appear again until in Feller's article [7] in 1952, which was concerned with deriving the Fokker-Planck equation, occuring in diffusion processes. Feller's equation has the specific form

$$V_t(t,x) = [(a(x)V(t,x))_x - b(x)V(t,x)]_x + \frac{\tau}{\sigma}V_2(t)\tilde{p}(x),$$

$$V_1'(t) = \frac{p_1}{\sigma} V_2(t),$$

$$V_2'(t) = -\frac{p_2}{\sigma} V_2(t) - \lim_{x\to r_2} [(a(x)V(t,x))_x - b(x)V(t,x)],$$

or when $\sigma = 0$,

$$V_t(t,x) = [(a(x)V(t,x))_x - b(x)V(t,x)]_x - p_2^{-1}\tilde{p}(x) \lim_{x\to r_2} [(a(x)V(t,x))_x - b(x)V(t,x)]$$

The boundary terms may be interpreted as mass on the boundary.

Phillips [15], in 1959 is a discussion of dissipative operators, used the following an example of a maximal dissipative operator: He defined the operator L by letting $Ly = y_x - y + y(0)h$, where the domain of L is $[y;\ y_x,\ y \in L^2(0,1)$ and $y(1) = 0]$.

More recently in a number of articles [9], [12], [5], [8] differential-boundary equations have been defined as adjoints to ordinary differential equations

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2. This reasearch was supported in part by NASA Grant NGR 39-009-041.

with integral boundary conditions.

In addition, the author has shown [10], [11], [12], [13] that in a Hilbert space the adjoint of a differential operator under integral boundary conditions is a differential-boundary operator. Further, these systems have been generalized to a situation where the adjoint of a differential-boundary system is itself a differential-boundary system.

The purpose of this article is to reduce a differential-boundary system to an ordinary differential system with end point boundary conditions, and to show how to eliminate some of the parameters which must be introduced in some representations.

2. <u>The Removal of Interface Conditions</u>. Interface conditions are boundary conditions applied at points interior to the interval under discussion. At such points the functions may be either continuous or be required to have some sort of finite discontinuity.

Let us consider a finite interval $[a,b]$ which is subdivided into m intervals by $a_1, \dots a_{m-1}$. Thus $a = a_o < a_1 < \dots < a_{m-1} < a_m = b$. Let Y be an $n \times 1$ vector. We shall consider boundary conditions of the form

$$M_i Y = \int_a^b K_i Y \, dx + \sum_{j=o}^{m} [A_{ij} Y(a_j +) + B_{ij} Y(a_j -)] = 0, \ i = 1,2,\dots,k, \text{ where } K_i$$

are $n \times n$ matrices, A_{ij} and B_{ij} are $n \times n$ matrices of constants, $Y(a_j +)$ and $Y(a_j -)$ denote the limits of $Y(x)$ as x approaches a_j from above and below. We assume that $A_{im} = 0$ and $B_{io} = 0$.

We define an operator L by letting $LY = Y' + PY$, where P is an $n \times n$ matrix. It was shown in [13] that if the domain of L is a suitably restricted subset of a Hilbert space satisfying $M_i Y = 0$, $i = 1,\dots,k$, the adjoint of L

is no longer a differential operator. It is advisable at this stage, therefore, to extend L in the following manner. Let C_{ij} and D_{ij}, $i = 1,\ldots,\ell$ be $n \times n$ constant matrices and let H_i, $i = 1,\ldots,\ell$ be $n \times n$ matrices. We define L_b (b for boundary) by $L_b Y = Y' + PY + \sum_{i=1}^{\ell} H_i \sum_{j=o}^{m} [(C_{ij}Y(a_j +) + D_{ij}Y(a_j -)]$ where, again, $C_{im} = 0$, $D_{io} = 0$.

Let us consider the system $\mathcal{S}$:

$$L_b Y = 0,$$

$$M_i Y = 0, \quad i = 1,\ldots,k.$$

It was shown in [13] that the proper adjoint system $\mathcal{S}^*$ is of the form:

$$L_b^* Z = - Z' + P^*Z - \sum_{i=1}^{k} K_i^* \phi_i(Z) = 0,$$

$$- Z(a_j -) + \sum_{\ell=1}^{k} B_{ij}^* \phi_i(Z) - \sum_{i=1}^{\ell} D_{ij}^* \int_a^b H_i^*Z \, dx = 0,$$

$$Z(a_{j-1} +) + \sum_{i=1}^{k} A_{ij-1} \phi_i(Z) - \sum_{i=1}^{\ell} C_{ij-1} \int_a^b H_i^*Z \, dx = 0,$$

$j = 1,2,\ldots,m$.

We wish to reduce the systems $\mathcal{S}$, $\mathcal{S}^*$ to a more manageable form.

We let $I_j = [a_{j-1}, a_j]$, $j = 1, \ldots, m$. We denote by $\mathcal{Y}$ the $nm \times 1$ vector

$\mathcal{Y} = \begin{pmatrix} Y(I_1) \\ \vdots \\ Y(I_m) \end{pmatrix}$, where the first n components are evaluated in I_1, the next n in I_2, etc. We let

$$L\mathcal{Y} = \mathcal{Y}' + \mathcal{P}\mathcal{Y}, \quad \text{where } \mathcal{P} = \begin{pmatrix} P(I_1) & 0 & \ldots & 0 \\ 0 & P(I_2) & \ldots & 0 \\ \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & P(I_m) \end{pmatrix}.$$

If

$$\mathcal{A} = \begin{pmatrix} A_{1o} & \cdots & A_{1m-1} \\ \cdot & \cdots & \cdot \\ A_{ko} & & A_{km-1} \end{pmatrix}, \qquad \mathcal{B} = \begin{pmatrix} B_{11} & \cdots & B_{1m} \\ \cdot & \cdots & \cdot \\ B_{k1} & & B_{km} \end{pmatrix},$$

$$\mathcal{K} = \begin{pmatrix} K_1(I_1) & \cdots & K_1(I_m) \\ \cdot & \cdots & \cdot \\ K_k(I_1) & \cdots & K_k(I_m) \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} H_1(I_1) & \cdots & H_\ell(I_1) \\ \cdot & \cdots & \cdot \\ H_1(I_m) & \cdots & H_\ell(I_m) \end{pmatrix},$$

$$\mathcal{C} = \begin{pmatrix} C_{io} & \cdots & C_{1m-1} \\ \cdot & \cdots & \cdot \\ C_{\ell o} & \cdots & C_{\ell m-1} \end{pmatrix}, \qquad \mathcal{D} = \begin{pmatrix} D_{11} & \cdots & D_{1m} \\ \cdot & \cdots & \cdot \\ D_{\ell 1} & \cdots & D_{\ell m} \end{pmatrix},$$

then $\mathcal{S}$ can be written as

$$L_b \mathcal{Y} = \mathcal{Y}' + \mathcal{P}\mathcal{Y} + \mathcal{H}\,[\mathcal{C}\mathcal{Y}(A) + \mathcal{D}\mathcal{Y}(B)] = 0,$$
$$\mathcal{A}\mathcal{Y}(A) + \mathcal{B}\mathcal{Y}(B) + \int_A^B \mathcal{K}(X)\mathcal{Y}(X)dX = 0,$$

where A consists of the n-tuple $(a_o+, a_1+, \ldots, a_{m-1}+)$, B is $(a_1-, a_2-, \ldots, a_m-)$, and X is $(x \in I_1, x \in I_2, \ldots, x \in I_m)$.

If $\mathcal{Z} = \begin{pmatrix} Z(I_1) \\ \vdots \\ Z(I_m) \end{pmatrix}$ and $\Phi(\mathcal{Z}) = \begin{pmatrix} \phi_1(Z) \\ \vdots \\ \phi_k(Z) \end{pmatrix}$, $\mathcal{S}^*$ can be written as

$$L_b^* \mathcal{Z} = -\mathcal{Z}' + \mathcal{P}^*\mathcal{Z} - \mathcal{K}^*\,\Phi(\mathcal{Z}) = 0,$$
$$-\mathcal{Z}(A) = \mathcal{A}^*\,\Phi(\mathcal{Z}) - \mathcal{C}^* \int_A^B \mathcal{H}^*(X)\mathcal{Z}(X)dX,$$
$$\mathcal{Z}(B) = \mathcal{B}^*\,\Phi(\mathcal{Z}) - \mathcal{D}^* \int_A^B \mathcal{H}^*(X)\mathcal{Z}(X)dX.$$

Both these forms do not involve interior boundary points. We might remark that the intervals I_j, $j = 1,\ldots,m$ can be parameterized in terms of a single variable t. Mansfield [14] has done this earlier under interior point conditions only.

We have therefore proved

Theorem 2.1. Differential-boundary systems and their adjoints involving interior point conditions may be reduced to such a system involving only end point conditions.

It is an easy computation to show

$$\int_A^B [\mathfrak{z}^*(L_b\mathfrak{y}) - (L_b^*\mathfrak{z})^*\mathfrak{y}]dX = 0.$$

3. Removal of the Parameter Φ. We assume that the boundary forms $\mathcal{A}\mathfrak{y}(A) + \mathcal{B}\mathfrak{y}(B)$ and $\mathcal{C}\mathfrak{y}(A) + \mathfrak{D}\mathfrak{y}(B)$ are linearly rowwise independent. (Otherwise in what follows the operators L_b also induce an integral operator with separable kernel, which, although easily handled, does not interest us. Green's formula for L (not L_b) yields

$$\begin{aligned}\int_A^B [\mathfrak{z}^*(L\mathfrak{y}) - (L^*\mathfrak{z})^*\mathfrak{y}]dx &= [\mathfrak{z}^*(A)\tilde{\mathcal{A}}^* + \mathfrak{z}^*(B)\tilde{\mathcal{B}}^*][\mathcal{A}\mathfrak{y}(A) + \mathcal{B}\mathfrak{y}(B)]\\ &+ [\mathfrak{z}^*(A)\tilde{\mathcal{C}}^* + \mathfrak{z}^*(B)\tilde{\mathfrak{D}}^*][\mathcal{C}\mathfrak{y}(A) + \mathfrak{D}\mathfrak{y}(B)]\\ &+ [\mathfrak{z}^*(A)\tilde{\mathfrak{E}}^* + \mathfrak{z}^*(B)\tilde{\mathfrak{F}}^*][\mathfrak{E}\mathfrak{y}(A) + \mathfrak{F}\mathfrak{y}(B)],\end{aligned}$$

where $\mathfrak{E}\mathfrak{y}(A) + \mathfrak{F}\mathfrak{y}(B)$ completes the number of independent boundary forms, and the $\mathfrak{z}$ terms are the appropriate complimentary forms. It is an easy computation to see that

Theorem 3.1. The proper adjoint system for $\mathfrak{S}$, $\mathfrak{S}_1^*$, is

$$\begin{aligned} L_b^*\mathfrak{z} = L^*\mathfrak{z} - \mathcal{K}[\tilde{\mathcal{A}}\mathfrak{z}(A) + \tilde{\mathcal{B}}\mathfrak{z}(B)] &= 0,\\ \tilde{\mathcal{C}}\mathfrak{z}(A) + \tilde{\mathfrak{D}}\mathfrak{z}(B) + \int_A^B \mathcal{H}^*(X)\mathfrak{z}(X)dX &= 0,\\ \tilde{\mathfrak{E}}\mathfrak{z}(A) + \tilde{\mathfrak{F}}\mathfrak{z}(B) &= 0.\end{aligned}$$

It can be proved that in an appropriate Hilbert space $\mathfrak{S}_1^*$ is the adjoint system.

As an example, it is easy to show that in $L^2(a,b)$ when $L_y = a_n y + a_{n-1}y' + \dots + a_o y^{(n)}$, $L^*z = \{\bar{a}_n z - [(\bar{a}_{n-1}) - [(\bar{a}_{n-2}z) - \dots - (\bar{a}_o z)']'\dots]'\}$, $a_o(x) \neq 0$, $a \leq x \leq b$, and

$$\int_a^b [\bar{z}(Ly) - (\overline{L^*z})y]dx = \bar{V}_1(z)U_1(y) + \dots + \bar{V}_n(z)U_n(z),$$

if $\mathcal{S}$ consists of

$$Ly + \sum_{i=1}^{m} K_i(x)U_i(y) = 0,$$

$$\int_a^b K_i y \, dx + U_i(y) = 0, \; i = m+1, \dots, n,$$

then $\mathcal{S}^*$ is

$$L^*z - \sum_{i=m+1}^{n} \bar{K}_i \, V_i(z) = 0,$$

$$\int_a^b \bar{K}_i z \, dx + V_i(z) = 0, \; i = 1, \dots, m.$$

4. Reduction to End Point Boundary Conditions. Not only is it possible to eliminate interior boundary conditions and the parameter Φ, it is also possible to write $\mathcal{S}$ as an ordinary differential system with end point boundary conditions. This was done recently by Jones [8] when $\mathcal{S}$ involved only a differential equation. Since interior points have been taken care of, let us consider only end point conditions and integrals.

$\mathcal{S}$ then consists of

$$Y' + PY + H[CY(a) + DY(b)] = 0, \; AY(a) + BY(b) + \int_a^b KYdx = 0,$$

where we may assume that Y is an $n \times 1$ matrix, P is $n \times n$, H is $n \times m$, C and D are $m \times n$, A, B and K are $p \times n$ matrices.

We let $U' = KY$, $U(a) = AY(a)$. Thus $U(x) = \int_a^x KY\,dx + U(a) = \int_a^x KY\,dx + AY(a)$ is a $p \times 1$ matrix. Further we let $S = CY(a) + DY(b)$. S is an $m \times 1$ matrix. $\mathcal{S}$ is now equivalent to the system

$$\begin{pmatrix} Y \\ U \\ S \end{pmatrix}' = \begin{bmatrix} -P & 0 & -H \\ K & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} Y \\ U \\ S \end{pmatrix},$$

$$\begin{bmatrix} A & -I & 0 \\ 0 & 0 & 0 \\ C & 0 & -\frac{1}{2}I \end{bmatrix} \begin{pmatrix} Y(a) \\ U(a) \\ S(a) \end{pmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ B & I & 0 \\ D & 0 & -\frac{1}{2}I \end{bmatrix} \begin{pmatrix} Y(b) \\ U(b) \\ S(b) \end{pmatrix} = 0 \quad .$$

According to the standard rules for finding adjoints, the adjoint system should be

$$\begin{pmatrix} Z \\ V \\ T \end{pmatrix}' = - \begin{bmatrix} -P^* & K^* & 0 \\ 0 & 0 & 0 \\ -H^* & 0 & 0 \end{bmatrix} \begin{pmatrix} Z \\ V \\ T \end{pmatrix},$$

$$\begin{bmatrix} I & A^* & C^* \\ 0 & 0 & -D^* \\ 0 & 0 & I \end{bmatrix} \begin{pmatrix} Z(a) \\ V(a) \\ T(a) \end{pmatrix} + \begin{bmatrix} 0 & 0 & -C^* \\ I & -B^* & D^* \\ 0 & 0 & I \end{bmatrix} \begin{pmatrix} Z(b) \\ V(b) \\ T(b) \end{pmatrix} = 0$$

This should be equivalent to the original form for $\mathcal{S}^*$. In fact, it is, as an easy computation shows.

We note that in $\mathcal{S}'$, Y is the dependent variable, U is a "boundary condition" variable, while S is a parameter (constant). In $\mathcal{S}^*$, however, although Z is the dependent variable, V is the parameter and T is the "boundary condition" variable. The last two components switch roles. As a result, such systems cannot be self-adjoint in the classical sense of Lagrange (see Coddington and Levinson [4]), but only in the symmetrizable sense of Bliss [2], [3] and Reid [16], [17].

In conclusion, we state

Theorem 4.1. Differential-boundary systems and their adjoints may be reduced to ordinary differential systems with end point boundary conditions.

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